Automorphisms of Engel Manifolds

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Introduction

What are Engel manifolds?

Let E be a 4-manifold. An *Engel structure* on E is a completely nonintegrable distribution whose rank 2.

F. Engel (1889,Darboux's theorem for Engel manifolds)

Let $(E, \mathcal{D})$ be an Engel manifold, and let $p \in E$. Then, there exists a chart $(U; x, y, z, w)$ with $p \in U$ such that
$\mathcal{D}|_U = Ker(dy - zdx) \cap Ker(dz - wdx)$.

Introduction

What are Engel manifolds?

R. Montgomery(1993)

A germ of rank k "good" distribution on a n-manifold is "stable"
$\Rightarrow dim(G_{k,n}) = k(n-k) \leq n$
$\Leftrightarrow k = 1$,
or, $k = n-1$,
or, $(n,k) = (4,2)$
or, tirivial (i.e. $k = 0, n$)

- The word "good" means that it admits a k-frame generating a finie dimensional Lie algebra.
- The word "stable" means that the distribution as section of a Grassmannian bundle is stable.

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$\Rightarrow dim(G_{k,n}) = k(n-k) \leq n$
$\Leftrightarrow k = 1$,
or, $k = n - 1$, ←contact, even contact
or, $(n, k) = (4, 2)$ ←Engel
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Introduction

Previous Work

Theorem (1999, R. Montgomery)

There is an Engel manifold such that the automorphism group is 1 dimensional at most if it is a Lie group.

Question (ref. AIM Problem Lists)

Is there an Engel manifold with trivial automorphism group?

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> *There is an Engel manifold such that the automorphism group is 1 dimensional at most if it is a Lie group.*

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$\longrightarrow$ The answer is "Yes".(Mitsumatsu, Y.)

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> **Theorem (1999, R. Montgomery)**
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> **Question (ref. AIM Problem Lists)**
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> Is there an Engel manifold with trivial automorphism group?

$\longrightarrow$ The answer is "Yes".(Mitsumatsu, Y.)
However, the following question is open.

> **Question (Mitsumatsu)**
>
> Is there an closed Engel manifold with trivial automorphism group?

Engel structures

Definition 1.1

Let E be a 4-manifold. An *Engel structure* on E is a smooth rank 2 distribution $\mathcal{D} \subset TE$ with following condition:

$$\mathcal{D}^2 \overset{\text{def}}{:=} \mathcal{D} + [\mathcal{D}, \mathcal{D}] \text{ has rank 3, and } \mathcal{D}^3 \overset{\text{def}}{:=} \mathcal{D}^2 + [\mathcal{D}^2, \mathcal{D}^2] \text{ has rank 4.}$$

The pair $(E, \mathcal{D})$ is called an *Engel manifold*.
Let $(E_1, \mathcal{D}_1), (E_2, \mathcal{D}_2)$ be Engel manifolds. An *Engel morphism* $f : (E_1, \mathcal{D}_1) \rightarrow (E_2, \mathcal{D}_2)$ is a local diffeomorphism $f : E_1 \rightarrow E_2$ with $df(\mathcal{D}_1) \subset \mathcal{D}_2$. (An *contact morphism* is similar.)

Propositoin 1.2 (R. Montgomery)

Let $(E, \mathcal{D})$ be an Engel manifold. Then, there exists a unique rank 1 distribution $\overset{\exists !}{}\mathcal{L} \subset \mathcal{D}$ such that $[\mathcal{L}, \mathcal{D}^2] \subset \mathcal{D}^2$.

- The above $\mathcal{L}$ is called the *characteristic foliation* of $(E, \mathcal{D})$.

Example 1.3

$E \overset{\text{def}}{=} J^2(1,1) \cong \mathbb{R}^4 \ni (x, y, \dot{y}, \ddot{y})$

$\mathcal{D} \overset{\text{def}}{=} Ker(\alpha) \cap Ker(\beta) \, (\alpha \overset{\text{def}}{=} dy - \dot{y}dx, \beta \overset{\text{def}}{=} d\dot{y} - \ddot{y}dx)$

$\longrightarrow (E, \mathcal{D})$ is an Engel manifold.

Then,

$\mathcal{L} = \langle \partial_{\ddot{y}} \rangle = Ker(\alpha) \cap Ker(\beta) \cap Ker(dx)$ is the characteristic foliation.

And,

$\mathcal{D} = \langle \partial_{\ddot{y}}, X \rangle = Ker(\alpha) \cap Ker(\beta) \, (X \overset{\text{def}}{=} \partial_x + \dot{y}\partial_y + \ddot{y}\partial_{\dot{y}}),$

$\mathcal{E} \overset{\text{def}}{=} \mathcal{D}^2 = \langle \partial_{\ddot{y}}, X, \partial_{\dot{y}} \rangle = Ker(\alpha).$

from Engel to contact

Any 3-dimensional submanifold $M \subset E$ intersecting transversally the characteristic foliation has a contact structure $TM \cap \mathcal{D}^2$.

Any vector field tangent to the characteristic foliation $\mathcal{L}$ preserves the "even contact structure" $\mathcal{D}^2$.

In particular, any holonomy of $\mathcal{L}$ is a germ of a contact morphism.

So the leaf space $E/\mathcal{L}$ has a contact structure $\mathcal{D}^2/\mathcal{L}$.

We want to research the relation between an Engel structure and the contact structure, but the leaf space is not necessarily a manifold. (In general, it is a Lie groupoid.)

Theorem 1.4 ((Corollary of) Reeb Stability Theorem)

Let X be a manifold and let $\mathcal{F}$ be a foliation on X.

1. *If all leaves of $\mathcal{F}$ is compact and all holonomy groups of that are finite, then the leaf space $X/\mathcal{F}$ is an orbifold.*

2. *As above, if all holonomy groups are trivial, then the leaf space $X/\mathcal{F}$ is a manifold.*

Definition 1.5 (Y.)

Let $(E, \mathcal{D})$ be an Engel manifold, and let $\mathcal{L}$ be the characteristic foliation of $(E, \mathcal{D})$.

1. If the leaf space $E/\mathcal{L}$ is an orbifold, then we says that $(E, \mathcal{D})$ has the *proper* characteristic foliation.

2. If the leaf space $E/\mathcal{L}$ is a manifold, then we says that $(E, \mathcal{D})$ has the *trivial* characteristic foliation.

Propositoin 1.6 (R. Montgomery)

Let $(E, \mathcal{D})$ be an Engel manifold that has the trivial characteristic foliation, let M be the leaf space of the characteristic foliation, and let $\pi : E \to M$ be the quotient map. Then, $\xi \overset{\mathrm{def}}{=} d\pi(\mathcal{D}^2)$ is well-defined, and it is a contact structure on M. Moreover, $(E, \mathcal{D}) \mapsto (M, \xi)$ is functorial.

Let (M, ξ) be a contact 3-manifold, and let

$$E = \mathbb{P}(\xi) \stackrel{\text{def}}{=} \coprod_{x \in M} \mathbb{P}(\xi_x) = (\xi - 0)/\mathbb{R}^{\times}.$$

Let $\pi : E \to M$ is the projective map. We define a rank 2 distribution $\mathcal{D}$ on E in the following way: For each $l \in E$ with $\pi(l) = x$, $l \subset \xi_x$ is a line that cross the origin. We define $\mathcal{D}_l \stackrel{\text{def}}{=} d\pi_l^{-1}(l) \subset T_l E$.

Similarly, let $E' = \mathbb{S}(\xi) \stackrel{\text{def}}{=} (\xi - 0)/\mathbb{R}_{>0}$. Then E' is a 2-covering on E. We define a rank 2 distribution $\mathcal{D}'$ on E' as the pull-back of $\mathcal{D}$.

Propositoin 1.7 (R. Montgomery)

The above $\mathcal{D}, \mathcal{D}'$ is an Engel structure on E, E'. Moreover, $(M, \xi) \mapsto (E, \mathcal{D})$ and $(E', \mathcal{D}')$ are functorial.

the Cartan prolongation

Definition 1.8

The above $(E, \mathcal{D})$ is called *Cartan prolongation*, and we write this $\mathbb{P}(M, \xi)$. The above $(E', \mathcal{D}')$ is called *oriented Cartan prolongation*, and we write this $\mathbb{S}(M, \xi)$.

Any Cartan prolongation has the trivial characteristic foliation. In fact, the Cartan prolongation is the "minimal" object of such Engel manifolds.

the developing map

Let $(E, \mathcal{D})$ be an Engel manifold that has the trivial characteristic foliation, and let (M, ξ) be the leaf space of the characteristic foliation. Let $\pi : E \to M$ be the quotient map. We define $\phi : E \to \mathbb{P}(\xi)$ as $\phi(e) \overset{\mathrm{def}}{=} d\pi(\mathcal{D}_e) \subset \xi_{\pi(e)}$.

> ## Propositoin 1.9 (R. Montgomery)
>
> *The above ϕ is an Engel morphism $(E, \mathcal{D}) \to \mathbb{P}(M, \xi)$. Moreover, this satisfies the universality: For any contact 3-manifold (N, ν) and any Engel morphism $\psi : (E, \mathcal{D}) \to \mathbb{P}(N, \nu)$, there exists a unique contact morphism $\tilde{\psi} : (M, \xi) \to (N, \nu)$ such that $\psi = \mathbb{P}(\tilde{\psi}) \circ \phi$.*

- The above ϕ is called the *developing map* associated to $(E, \mathcal{D})$.
- The functor $\mathbb{P}$ is fully faithful.

extend to orbifolds

The above discussion can be generalized to an Engel manifold with proper characteristic foliation in the obvious way. That is, if $E/\mathcal{L}$ is an orbifold, then $E/\mathcal{L}$ has a contact structure, and then E has the developing map. However, the Cartan prolongation of a contact 3-orbifold is an Engel orbifold in general.

Question

When is the Cartan prolongation of a contact 3-orbifold an Engel manifold?

Main result 1

Theorem 2.1 (Y.)

Let (Σ, ξ) be a contact 3-orbifold.

1. The Cartan prolongation of (Σ, ξ) is a manifold if and only if (Σ, ξ) is *positive*, and $|G_x|$ is odd for all $x \in \Sigma$, where G_x is the isotropy group at x.

2. The oriented Cartan prolongation of (Σ, ξ) is a manifold if and only if (Σ, ξ) is *positive*.

In fact, all Engel manifolds obtained as the Cartan prolongation of a "space" with contact structure are obtained as above. This can be shown by using Lie groupoid theory.

orbifolds and contact structures

- Recall, a *manifold* is a topological space to be locally Euclidean.
- An *orbifold* is a topological space to be locally Euclidean with a finite group action.

Definition 2.2

Let Σ be an orbifold, and let $\{(V_\lambda \subset \mathbb{R}^n, G_\lambda \curvearrowright V_\lambda, p_\lambda : V_\lambda \to \Sigma)\}_{\lambda \in \Lambda}$ be an orbifold atlas on Σ, where $\phi_{\lambda\mu}$ is the transformation map for $\lambda, \mu \in \Lambda$.
A *contact structure* on Σ is a family $\xi = \{\xi_\lambda\}_{\lambda \in \Lambda}$ of contact structures on each V_λ such that all $G_\lambda \curvearrowright V_\lambda$ are contact actions, and all $\phi_{\lambda\mu}$ are contact morphisms. The pair of an orbifold and a contact structure is called a *contact orbifold*.
Similarly, we define an *Engel structure* on an orbifold and an *Engel orbifold*.

Example 2.3

$\xi_{std} \overset{\text{def}}{=} Ker(dz + xdy - ydx)$ is a contact structure on $\mathbb{R}^3$. We define $\phi_n, \psi : \mathbb{R}^3 \to \mathbb{R}^3$ as $\phi_n(x, y, z) \overset{\text{def}}{=}$ $(x\cos(\frac{2\pi}{n}) - y\sin(\frac{2\pi}{n}), x\sin(\frac{2\pi}{n}) + y\cos(\frac{2\pi}{n}), z), \psi(x, y, z) \overset{\text{def}}{=} (x, -y, -z)$. Then, $G_{n,std} \overset{\text{def}}{=} <\phi_n, \psi> \curvearrowright (\mathbb{R}^3, \xi_{std})$ and $H_{n,std} \overset{\text{def}}{=} <\phi_n> \curvearrowright (\mathbb{R}^3, \xi_{std})$ are contact actions of finite groups. These are called *standard models*. Then, $(\mathbb{R}^3, \xi_{std})/G_{n,std}$ and $(\mathbb{R}^3, \xi_{std})/H_{n,std}$ are contact orbifolds.

Theorem 2.4 (Darboux's theorem for contact 3-orbifolds)

Let (Σ, ξ) be a contact 3-orbifold, and let $x \in \Sigma$. Then, there exists an orbifold chart (V, G, p) around x such that (V, G) is isomorphic to an open neighborhood of $0 \in (\mathbb{R}^3, H)$ where $H = G_{n,std}$ or $H = H_{n,std}$.

Definition 2.5

Let Σ be an orbifold, and let $x \in \Sigma$. Take an orbifold chart (V, G, p) with $x \in p(V) \subset \Sigma$, and take a point $\tilde{x} \in p^{-1}(x)$. Then, $G_x \stackrel{\mathrm{def}}{=} \{\sigma \in G \mid \sigma \tilde{x} = \tilde{x}\}$ is called the *isotropy group* at x.

Definition 2.6

A contact orbifold (Σ, ξ) is *positive* if all $G_x \curvearrowright \xi_x$ preserve the orientation.

- $(\mathbb{R}^3, \xi_{std})/G_{n,std}$ is not positive.
- $(\mathbb{R}^3, \xi_{std})/H_{n,std}$ is positive.

Main lemma

Because of Darboux's theorem for contact 3-orbifolds, the main result follows the following lemma.

Lemma 2.7

1. *The actions $G_{n,std} \curvearrowright \mathbb{P}(\xi_{std})$, $G_{n,std} \curvearrowright \mathbb{S}(\xi_{std})$ are not free.*
2. *The action $H_{n,std} \curvearrowright \mathbb{P}(\xi_{std})$ is free if and only if n is odd.*
3. *The action $H_{n,std} \curvearrowright \mathbb{S}(\xi_{std})$ is free for any n.*

Main result 2

The composition of the two functors

$$\{Engel\} \longrightarrow \{Contact\} \longrightarrow \{Engel\}$$

$$\cup \qquad\qquad \cup \qquad\qquad \cup$$

$$(E, \mathcal{D}) \longmapsto (E/\mathcal{L}, \mathcal{D}^2/\mathcal{L}) \longmapsto \mathbb{P}(E/\mathcal{L}, \mathcal{D}^2/\mathcal{L})$$

induces a group morphism

$$\Phi : Aut(E, \mathcal{D}) \to Aut(E/\mathcal{L}, \mathcal{D}^2/\mathcal{L}) \cong Aut(\mathbb{P}(E/\mathcal{L}, \mathcal{D}^2/\mathcal{L}))$$

Theorem 2.8 (Y.)

Let $(E, \mathcal{D})$ be a connected Engel manifold. Suppose that $\mathbb{P}(E/\mathcal{L}, \mathcal{D}^2/\mathcal{L})$ is a manifold. If the developing map $\phi : (E, \mathcal{D}) \to \mathbb{P}(E/\mathcal{L}, \mathcal{D}^2/\mathcal{L})$ is not covering, then the above group morphism Φ is injective.

Previous Work

Theorem (1999, R. Montgomery)

Let $(E, \mathcal{D})$ be an Engel manifold, and $\Sigma \subset E$ be a global section of the characteristic foliation $\mathcal{L}$. (i.e. Σ is a 3-submanifold that intersects $\mathcal{L}$ transversely.) For Engel automorphisms $f, g : (E, \mathcal{D}) \to (E, \mathcal{D})$, f and g coincide on E iff they coincide on Σ.

In Montgomery's theorem, the Engel isomorphisms are controlled on a global section.

In our theorem, the Engel isomorphisms are controlled on the infinity points of the leaves.

Main lemma

Lemma 2.9

Let $(E, \mathcal{D})$ be a connected Engel manifold. Suppose that $\mathbb{P}(E/\mathcal{L}, \mathcal{D}^2/\mathcal{L})$ is a manifold.

1. *If there exists a leaf L of $\mathcal{L}$ such that $\phi|_L : L \to \mathbb{P}(\mathcal{D}^2/\mathcal{L})_L$ is not covering, then the above group morphism Φ is injective.*

2. *If, for any leaf L of $\mathcal{L}$, $\phi|_L : L \to \mathbb{P}(\mathcal{D}^2/\mathcal{L})_L$ is covering, then ϕ is covering*

proof of 1.

Suppose $\Phi(f) = \Phi(g)$ for $f, g \in Aut(E, \mathcal{D})$.
Because $\phi|_L$ is not covering, we can identify $\phi|_L$ with
$(a, b) \to \mathbb{R}/\mathbb{Z} \, (a \neq -\infty)$.

$$
\begin{array}{ccc}
(c, d) & \xrightarrow{\ \phi\ } & \mathbb{R}/\mathbb{Z} \\
f \Big\uparrow\Big\uparrow g & & \Big\uparrow \Phi(f) = \Phi(g) \\
(a, b) & \xrightarrow[\ \phi\ ]{} & \mathbb{R}/\mathbb{Z}
\end{array}
$$

Then there exists $\epsilon > 0$ such that $f(a + \epsilon) = g(a + \epsilon)$.
Because E is connected, we obtain $f = g$.

$\square$

proof of 2.

Take any leaf L of $\mathcal{L}$, and let $L' \stackrel{\text{def}}{=} \mathbb{P}(\mathcal{D}^2/\mathcal{L})_L$.

There exists a nonsingular vector fiald X realizing the holonomy on L', defined on a open neighborhood U. (By Reeb Stability Theorem.)

Let $\tilde{X}$ be a pull-back of X by ϕ. Because any $\phi|_{L_0}$ is covering, $\tilde{X}$ is complete.

So, we can obtain $\phi|_{\phi^{-1}(U)} : \phi^{-1}(U) \to U$ is covering whose degree is the same as that of $\phi|_L$.

$\square$

construction of the answer of AIM's problem

Let $(E, \mathcal{D})$ be a connected Engel manifold. Suppose that $E/\mathcal{L}$ is a manifold.

Define $\sigma : E/\mathcal{L} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by $\sigma(L) \overset{\text{def}}{=} \min\{\#\phi^{-1}(y) \mid y \in \mathbb{P}(\mathcal{D}^2/\mathcal{L})_L\}$.

Call σ *twisting number function*.

Propositoin 2.10

For any $f \in Aut(E, \mathcal{D})$, the induced automorphism $E/\mathcal{L} \to E/\mathcal{L}$ preserves the twisting number function σ.

Example 2.11

$E \overset{\text{def}}{=} \mathbb{R}^4 \ni (x, y, z, \theta)$

$\mathcal{D} \overset{\text{def}}{=} \langle \partial_\theta, \cos(\frac{\theta}{2})X + \sin(\frac{\theta}{2})Y \rangle \, (X \overset{\text{def}}{=} \partial_x - y\partial_z, \, Y \overset{\text{def}}{=} \partial_y)$

$\quad \longrightarrow (E, \mathcal{D})$ is an Engel manifold.

Then, $\mathcal{L} = \langle \partial_\theta \rangle$ is the characteristic foliation.

And, $\mathcal{E} \overset{\text{def}}{=} \mathcal{D}^2 = \langle \partial_\theta, X, Y \rangle$.

$\quad \longrightarrow M \overset{\text{def}}{=} E/\mathcal{L} \cong \mathbb{R}^3 \ni (x, y, z), \, \xi \overset{\text{def}}{=} \mathcal{E}/\mathcal{L} \cong \langle X, Y \rangle$.

$M \times S^1 \cong \mathbb{P}(M, \xi); (\boldsymbol{x}, [\theta]) \mapsto \langle \cos(\frac{\theta}{2})X_{\boldsymbol{x}} + \sin(\frac{\theta}{2})Y_{\boldsymbol{x}} \rangle$.

$\phi : E \to M \times S^1; (\boldsymbol{x}, \theta) \mapsto (\boldsymbol{x}, [\theta])$ is identified with the developing map.

$\quad \longrightarrow$ The twisting number function σ is a constant function ∞.

Example 2.12

Fix a point $\boldsymbol{x}_0 \in \mathbb{R}^3$ and a number $n \in \mathbb{Z}_{\geq 0}$.

$E \overset{\text{def}}{=} \mathbb{R}^4 - \{\boldsymbol{x}_0\} \times ((-\infty, -n\pi] \cup [n\pi + \epsilon, \infty)) \, (\epsilon \in (0, 2\pi])$

$\mathcal{D} \overset{\text{def}}{=} \langle \partial_\theta, \cos(\frac{\theta}{2}) X + \sin(\frac{\theta}{2}) Y \rangle \, (X \overset{\text{def}}{=} \partial_x - y\partial_z, \, Y \overset{\text{def}}{=} \partial_y)$

$\longrightarrow (E, \mathcal{D})$ is an Engel manifold.

Then, $\mathcal{L} = \langle \partial_\theta \rangle$ is the characteristic foliation.

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$\longrightarrow M \overset{\text{def}}{=} E/\mathcal{L} \cong \mathbb{R}^3 \ni (x, y, z), \, \xi \overset{\text{def}}{=} \mathcal{E}/\mathcal{L} \cong \langle X, Y \rangle$.

Then, $M \times S^1 \cong \mathbb{P}(M, \xi); (\boldsymbol{x}, [\theta]) \mapsto \langle \cos(\frac{\theta}{2}) X_{\boldsymbol{x}} + \sin(\frac{\theta}{2}) Y_{\boldsymbol{x}} \rangle$.

$\phi : E \to M \times S^1; (\boldsymbol{x}, \theta) \mapsto (\boldsymbol{x}, [\theta])$ is identified with the developing map.

$\longrightarrow$ The twisting number function σ is the following:

$$\sigma(\boldsymbol{x}) = \begin{cases} n & (\boldsymbol{x} = \boldsymbol{x}_0) \\ \infty & (\textit{otherwise}) \end{cases}$$

Theorem 2.13 (Mitsumatsu, Y.)

There exists an Engel manifold with trivial automorphism group.

proof

Take a countable dense subset $Q = \{\boldsymbol{x}_n\}_{n=1}^{\infty} \subset \mathbb{R}^3$.

$E \stackrel{\text{def}}{=} \mathbb{R}^4 - \bigcup_n \{\boldsymbol{x}_n\} \times ((-\infty, -n\pi] \cup [n\pi + \epsilon, \infty)) \, (\epsilon \in (0, 2\pi])$

$\mathcal{D} \stackrel{\text{def}}{=} \langle \partial_\theta, \cos(\frac{\theta}{2})X + \sin(\frac{\theta}{2})Y \rangle \, (X \stackrel{\text{def}}{=} \partial_x - y\partial_z, \, Y \stackrel{\text{def}}{=} \partial_y)$

   $\longrightarrow (E, \mathcal{D})$ is an Engel manifold.

   $\longrightarrow$ The twisting number function σ is the following:

$$\sigma(\boldsymbol{x}) = \begin{cases} n & (\boldsymbol{x} = \boldsymbol{x}_n) \\ \infty & (otherwise) \end{cases}$$

For any $f \in Aut(E, \mathcal{D})$, the induced automorphism $\underline{f} : \mathbb{R}^3 \to \mathbb{R}^3$ is the identity on Q. Because $Q \subset \mathbb{R}^3$ is dense, $\underline{f}$ is the identity on $\mathbb{R}^3$. So f is the identity.

$\square$