Sheaf theoretic characterization of topological etale groupoids

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February 10th, 2020

Motivation

	as étale space	as functor
Sheaves (on X)	$E \to X$	$X_{top}^{op} \to \mathbf{Set}$
Étale groupoids (on $\mathcal{G}_0$)	$\mathcal{G}_1 \overset{s}{\underset{t}{\rightrightarrows}} \mathcal{G}_0$	?

$\rightsquigarrow$

we obtain some (trivial) ideas:

- "sheafifications of pre-groupoids"
- "enriched étale groupoids"

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Étale groupoids (on $\mathcal{G}_0$)	$\mathcal{G}_1 \overset{s}{\underset{t}{\rightrightarrows}} \mathcal{G}_0$	pseudogroup sheaves

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Previous work

1904(Cartan) a pseudogroup is defined

1932(Whitehead) the term “pseudogroup” appears

Definition 0.1

A *pseuedogroup* on a topological space X is a subgroupoid of the set of homeo. between open sets of X, satisfying *the sheaf property*.

Previous work

1958(Haeflier) from pseudogroups via étale groupoids

1998(Lawson) abstract pseuedogroups are equivalent to étale groupoids

2007(Resende) étale groupoids are equivalent to quantales

Definition 0.2

An *abstract pseuedogroup* is a complete and infinitely distributive inverse semigroup.

$\rightsquigarrow$ This is not sheaf theoretical!

§1 topological étale groupoids

Recall, a *groupoid* is a category such that any morphism is an isomorphism.
A *topological groupoid* is a groupoid $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, s, t, i, inv, comp)$ such that the set of objects $\mathcal{G}_0$ and the set of morphisms $\mathcal{G}_1$ are topological spaces, and that the structure maps (i.e. source map s, target map t, identities map i, inversion map inv and composition map $comp$) are continuous.
A topological groupoid is *étale* if the source map and the target map are local homeomorphisms.
Let X be a topological space. A topological groupoid on X is a topological groupoid $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ such that $\mathcal{G}_0 = X$.

sheaves associated with topological étale groupoids

Let X be a topological space. $X_{top} \overset{\text{def}}{=} \{U \subset X : open\}$.

$x \to y \overset{\text{def}}{\Leftrightarrow} \mathcal{N}(y) \subset \mathcal{N}(x)$ where $\mathcal{N}(x) \overset{\text{def}}{=} \{U \in X_{top} | x \in U\}$.

Let $\mathcal{G}$ be a topological étale groupoid on X.

$\hat{\mathcal{G}}(U, V) \overset{\text{def}}{=} \{f : U \to t^{-1}(V) | s \circ f = id\}$ for $U, V \in X_{top}$.

$\hat{\mathcal{G}}$ is a small category with including X_{top} (i.e. $X_{top} \subset \hat{\mathcal{G}}$) as a subcategory.

$\hat{\mathcal{G}}_x(V) \overset{\text{def}}{=} \varinjlim_{U \ni x} \hat{\mathcal{G}}(-, V)$ for $V \in X_{top}$ $x \in X$.

$\hat{\mathcal{G}}_x^y \overset{\text{def}}{=} \varprojlim_{V \ni y} \hat{\mathcal{G}}_x(V)$ for $x, y \in X$.

Propositoin 1.1

The above $\hat{\mathcal{G}}$ satisfies the following:

(1) $Ob(X_{top}) = Ob(\hat{\mathcal{G}})$.

(2.1) the natural projections $\hat{\mathcal{G}}_x^y \to \hat{\mathcal{G}}_x(V)$ are injections.

(2.2) $\coprod_{y \in V} \hat{\mathcal{G}}_x^y \to \hat{\mathcal{G}}_x(V)$ is surjection.

(2.3) $y \to z \Rightarrow \hat{\mathcal{G}}_x^y \subset \hat{\mathcal{G}}_x^z$.
$f_x \in \hat{\mathcal{G}}_x^y \Leftrightarrow f(x) \to y$ for any $f_x \in \hat{\mathcal{G}}_x(V)$.

(3) the presheaf $\hat{\mathcal{G}}(-, V)$ is a sheaf for each $V \in X_{top}$.

Remark 1.2

If X is a T_1 space (i.e. $x \to y \Rightarrow x = y$), all together the above conditions (2.1), (2.2) and (2.3) equivalents the following:

(2) $\hat{\mathcal{G}}_x(V) \cong \coprod_{y \in V} \hat{\mathcal{G}}_x^y$ for $V \in X_{top}$ and $x \in X$, where the canonical projections $\hat{\mathcal{G}}_x^y \to \hat{\mathcal{G}}_x(V)$ are identified with the canonical injections $\hat{\mathcal{G}}_x^y \to \coprod_{y \in V} \hat{\mathcal{G}}_x^y$.

§2 pseudogroup sheaves

Suppose that $\mathcal{C}$ is a small category such that $X_{top} \subset \mathcal{C}$, $Ob(X_{top}) = Ob(\mathcal{C})$. Then $\mathcal{C}(-, V) : X_{top}^{op} \subset \mathcal{C}^{op} \to \mathbf{Set}$ is a presheaf on X for $V \in X_{top}$.

$\mathcal{C}_x(V) \overset{\text{def}}{=} \varinjlim_{U \ni x} \mathcal{C}(-, V)$ for $V \in X_{top}$ $x \in X$.

$\mathcal{C}_x^y \overset{\text{def}}{=} \varprojlim_{V \ni y} \mathcal{C}_x(V)$ for $x, y \in X$.

The composition map of the category $\mathcal{C}$ induce $\mathcal{C}_y^z \times \mathcal{C}_x^y \to \mathcal{C}_x^z$.

Then, we can define the category $\mathcal{C}^\star$ by $Ob(\mathcal{C}^\star) \overset{\text{def}}{=} X$ and $\mathcal{C}^\star(x, y) \overset{\text{def}}{=} \mathcal{C}_x^y$.

Definition 2.1 (Y.)

Let X be a T_1 space. A *pre-pseudogroup* on X is a small category $\mathcal{C}$ with an embedding functor $X_{top} \subset \mathcal{C}$ satisfying the following:

(1) $Ob(X_{top}) = Ob(\mathcal{C})$.

(2) $\mathcal{C}_x(V) \cong \coprod_{y \in V} \mathcal{C}_x^y$ for $V \in X_{top}$ and $x \in X$, where the canonical projections $\mathcal{C}_x^y \to \mathcal{C}_x(V)$ are identified with the canonical injections $\mathcal{C}_x^y \to \coprod_{y \in V} \mathcal{C}_x^y$.

(4) the category $\mathcal{C}^\star$ is a groupoid.

A *pseudogroup sheaf* on X is a pre-pseudogroup on X satisfying the following:

(3) the presheaf $\mathcal{C}(-, V)$ is a sheaf for each $V \in X_{top}$.

Examples

Example 2.2

Let $Homeo^l_X(U, V) \overset{\text{def}}{=} \{f : U \to V : \textit{local homeo.}\}$ for $U, V \in X_{top}$. $Homeo^l_X$ satisfies the above conditions (1), (3) and (4). If X is T_1, then $Homeo^l_X$ is a pseudogroup sheaf on X.

Example 2.3

Let $\mathcal{F}$ be a sheaf of group on X. We define $\mathcal{C}$ as

$$\mathcal{C}(U, V) \overset{\text{def}}{=} \begin{cases} \mathcal{F}(U) & (U \subset V) \\ \emptyset & (U \not\subset V). \end{cases}$$

Then $\mathcal{C}$ is a pseudogroup sheaf on X.

underlying maps

Let X be a T_1 space, and let $\mathcal{C}$ be a pre-pseudogroup on X.
For $f \in \mathcal{C}(U, V)$, define $\bar{f} : U \to V$ in the following:

Take any point $x \in U$.
Then the germ f_x of f at x belong to $\mathcal{C}_x(V) \cong \coprod_{y \in V} \mathcal{C}_x^y$.
So there exists exactly one point $y \in V$ such that $f_x \in \mathcal{C}_x^y$.
Define $\bar{f}(x) \overset{\mathrm{def}}{=} y$.

Remark 2.4

$\overline{g \circ f} = \bar{g} \circ \bar{f}$ and $\bar{id} = id$.
Call $\bar{f}$ the *underlying map* of f.

Propositoin 2.5

The underlying map $\bar{f} : U \to V$ *is continuous.*

Proof.

Take any open set $V' \subset V$ and any point $x \in \bar{f}^{-1}(V')$. Then $f_x \in \mathcal{C}_x(V')$. So there exists an open neighborhood $U' \subset U$ of x such that $f|_{U'} \in \mathcal{C}(U', V')$. Namely, $x \in U' \subset \bar{f}^{-1}(V')$. □

Remark 2.6

Moreover, $\bar{f}$ is local homeomorphism because $\mathcal{C}^{\star}$ is a groupoid.

classical pseudogroups

Definition 2.7

A pseudogroup sheaf $\mathcal{C}$ on X is *concrete* if the functor $\mathcal{C} \to Homeo^l_X; f \mapsto \bar{f}$ is faithful.

Let $\mathcal{C}$ be a concrete pseudogroup on X. Then the subcategory of invertible morphisms of $\mathcal{C}$ is a *pseudogroup* in a classical sence.
Conversely, we can obtain a concrete pseudogroup in the new sence from any pseudogroup in the classical sence, by "sheafification".

from pseudogroup sheaves via étale groupoids

Let $s : E_{\mathcal{C}} \to X$ be an étale space associated with the pre-sheaf $\mathcal{C}(-, X)$.
Namely, $E_{\mathcal{C}} = \coprod_{x \in X} \mathcal{C}_x(X) \cong \coprod_{x,y \in X} \mathcal{C}_x^y$ and $s(f_x) = x$ for $f_x \in \mathcal{C}_x^y$.
In other words, $E_{\mathcal{C}} = Mor(\mathcal{C}^\star)$ is a set of morphisms of $\mathcal{C}^\star$, and s is a source map of $\mathcal{C}^\star$.

Propositoin 2.8

$\mathcal{C}^\star$ is a topological étale groupoid.

Proof.

We have to show that

- the target map $t : E_{\mathcal{C}} \to X$

is a local homeomorphism, and that

- the identities map $i : X \to E_{\mathcal{C}}$,
- the inversion map $inv : E_{\mathcal{C}} \to E_{\mathcal{C}}$, and
- the composition map $comp : E_{\mathcal{C}} \times_X E_{\mathcal{C}} \to E_{\mathcal{C}}$

are continuous.

$t = s \circ inv$ and $inv \circ inv = id$, so We have to proof that i, inv, and $comp$ are continuous.

$i(x) = (id_X)_x$, so i is continuous.

In the similar proof of the Prop 2.5, *inv* and *comp* are continuous. □

Main Result

Theorem 2.9 (Y.)

Let X be a T_1 space. There exists one-to-one correspondence between topological étale groupoids over X and pseudogroup sheaves on X.

Recall the sheafification of a presheaf. Let $\mathcal{F}$ be a presheaf, and let $S_x\mathcal{F}$ be the skyscraper sheaf supported at x with valued $\mathcal{F}_x$, where $\mathcal{F}_x$ is the stalk of $\mathcal{F}$ at x.
Let $\mathcal{F}^{\#} \stackrel{\text{def}}{=} \prod_x S_x\mathcal{F}$. Now, we define $\hat{\mathcal{F}}$ as the minimum subsheaf of $\mathcal{F}^{\#}$ including $Im[\mathcal{F} \to \mathcal{F}^{\#}]$. $\mathcal{F} \to \hat{\mathcal{F}}$ is the sheafification.

Definition 3.1

Let $\mathcal{C}$, $\mathcal{D}$ be pre-pseudogroups on X. A *morphism* $f : \mathcal{C} \to \mathcal{D}$ is a functor $f : \mathcal{C} \to \mathcal{D}$ preserving X_{top}. i.e. The following diagram is commutative:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \uparrow & & \uparrow \\ X_{top} & = & X_{top} \end{array}$$

Let $\mathcal{C}$ be a pre-pseudogroup on X, and let $\mathcal{C}^{\#}(U, V) \stackrel{\text{def}}{=} \mathcal{C}(U, V)^{\#}$ for each open sets $U, V \in X_{top}$.

The composition map $\mathcal{C}(V, W) \times \mathcal{C}(U, V) \to \mathcal{C}(U, W)$ induces $\mathcal{C}^{\#}(V, W) \times \mathcal{C}^{\#}(U, V) \to \mathcal{C}^{\#}(U, W)$.

$\mathcal{C}^{\#}$ is a pseudogroup sheaf on X.

Theorem 3.2 (Y.)

Let X be a T_1 space, and let $\mathcal{C}$ be a pre-pseudogroup on X. There exists a pseudogroup sheaf $\hat{\mathcal{C}}$ on X and a morphism $\mathcal{C} \to \hat{\mathcal{C}}$ satisfying universality: For any pseudogroup sheaf $\mathcal{D}$ on X and a morphism $\mathcal{C} \to \mathcal{D}$, there exists an unique morphism $\hat{\mathcal{C}} \to \mathcal{D}$ such that the following diagram is commutative;

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \hat{\mathcal{C}} \\ & \searrow & \downarrow \\ & & \mathcal{D} \end{array}$$

Proof.

Let $\hat{\mathcal{C}}$ be the minimum sub-pseudogroup sheaf of $\mathcal{C}^{\#}$ including $Im[\mathcal{C} \to \mathcal{C}^{\#}]$. $\mathcal{C} \to \hat{\mathcal{C}}$ satisfies the above property. □

Propositoin 3.3

$\hat{\mathcal{C}}(-, V)$ is the sheafification of $\mathcal{C}(-, V)$ for any $V \in X_{top}$.

Proof.

By stalk-wise discussion. □

Remark 3.4

These prooves depend on the following two properties:

- each functor $(-) \times A$ preserves any coproduct
- a morphism $f : \mathcal{F} \to \mathcal{G}$ between sheaves is an isomorphism if and only if all of the induced map $f_x : \mathcal{F}_x \to \mathcal{G}_x$ between stalks are isomorphisms

The first property holds in any Cartesian closed category.
The second property is known as *to have enough points* in *Topos Theory*. From this property, a lot of stalk-wise discussion can be justified. This property is not necessary for Theorem 3.2 to hold.

§4 enriched version

Let $\mathcal{E}$ be a complete and cocomplete category.
$\longrightarrow \mathcal{E}$ is a Cartesian monoidal category.

Recall, roughly speaking, a *$\mathcal{E}$-enriched category* is a category with $\mathcal{E}$-valued Hom functor.

$\mathcal{E}$ has an initial object $\emptyset$ and a terminal object $*$.
So, we can regard X_{top} as $\mathcal{E}$-enriched category. Directly, we define it in the following:

$$X_{top}(U, V) \overset{\text{def}}{=} \begin{cases} * & (U \subset V) \\ \emptyset & (U \not\subset V) \end{cases}$$

Definition 4.1

Let X be a T_1 space. An *$\mathcal{E}$-enriched pre-pseudogroup* on X is an $\mathcal{E}$-enriched category $\mathcal{C}$ with an $\mathcal{E}$-enriched functor $X_{top} \to \mathcal{C}$ satisfying the following:

(1) $Ob(X_{top}) = Ob(\mathcal{C})$.

(2) $\mathcal{C}_x(V) \cong \coprod_{y \in V} \mathcal{C}_x^y$ for $V \in X_{top}$ and $x \in X$, where the canonical projections $\mathcal{C}_x^y \to \mathcal{C}_x(V)$ are identified with the canonical injections $\mathcal{C}_x^y \to \coprod_{y \in V} \mathcal{C}_x^y$.

(4) the $\mathcal{E}$-enriched category $\mathcal{C}^\star$ is an $\mathcal{E}$-enriched groupoid.

A *$\mathcal{E}$-enriched pseudogroup sheaf* on X is a pre-pseudogroup on X satisfying the following:

(3) the presheaf $\mathcal{C}(-, V)$ is a sheaf for each $V \in X_{top}$.

Let $\mathcal{E}$ be a category satisfying the following:

- $\mathcal{E}$ is complete and cocomplete.
- filtered colimits in $\mathcal{E}$ are exact.
- *IPC-property* holds in $\mathcal{E}$.

$\longrightarrow$ Any $\mathcal{E}$-valued presheaf has a sheafification.
(cf. Kashiwara-Schapira, "Categories and Sheaves")

And suppose that the properties of Remark 3.4 hold in $\mathcal{E}$.
$\longrightarrow$ Any $\mathcal{E}$-enriched pre-pseudogroup has a sheafification.

Examples

Example 4.2

Set, the category of sets, satisfies the all above properties.
A **Set**-enriched pre-pseudogroup is just a pre-pseudogroup.

Example 4.3

Let G be a group.
G-**Set**, the category of sets with G-action, satisfies the all above properties.
So, any G-**Set**-enriched pre-pseudogroup has a sheafification.

Examples

Example 4.4

Let $\mathcal{S}$ be a small category.
Then, $PSh(\mathcal{S})$, the category of presheaves on $\mathcal{S}$, satisfies the all above properties.
In particular, **SSet**, the category of simplicial sets, satisfies the all above properties.

Example 4.5

Let $\mathcal{S}$ be a small site (i.e. a small category with a grothendieck topology).
And suppose that all of the coverings are finite.
Then, $Sh(\mathcal{S})$, the category of sheaves on $\mathcal{S}$, satisfies the all above properties.